Concrete univalent mathematics

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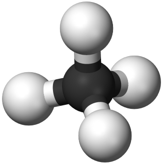
June 13, 2024



SYMMETRY

Am Anfang war die Symmetrie – In the beginning was symmetry!

Werner Heisenberg, *Der Teil und das Ganze: Gespräche im Umkreis der Atomphysik*, 1969, English translation, *Physics and Beyond*, 1971.

by

Marc Bezem
Ulrik Buchholtz
Pierre Cagne
Bjørn Ian Dundas
Daniel R. Grayson

What is univalent mathematics?

Some partial answers

- Univalent mathematics takes the notion of ∞-groupoid as primitive, rather than the notion of set in classical mathematics.
- Univalent mathematics is a foundation of mathematics that formalizes the common mathematical practice of considering isomorphic objects the same.
- Univalent mathematics is based on Martin-Löf's dependent type theory.
- Equality of any type in univalent mathematics can be fully characterized.
- Univalent mathematics is compatible with classical mathematics as well as various branches of constructive mathematics.

Types as ∞-groupoids

Identity types

Every type comes equipped with its identity type

$$- = - : A \to A \to \mathsf{Type}$$

Identity types are families of types inductively generated by

$$\mathsf{refl} : \prod_{x:A} x = x$$

Since the types $x = y$ of identifications from x to y are themselves types, they may possess many distinct elements. By the induction principle of identity types, it follows that every type has the structure of a weak ω-groupoid.

The homotopy interpretation

By default, the primitive objects of type theory are weak ω-groupoids, not sets

Truncation levels

Types can be organized by the complexity of their identity types. This is the hierarchy of truncation levels:

Truncation levels

- At the bottom of this hierarchy we have contractible types, i.e., types with an equivalence

$$X \simeq \mathbf{1}$$

The truncation level of contractible types is set to be -2.

- A type is said to be of truncation level $k+1$ if its identity types are of truncation level k.
- Types of truncation level -1 are *propositions*. Identity types of propositions are contractible.
- Types of truncation level 0 are *sets*. Identity types of sets are propositions.

Axiom K is the assertion that all types are sets in this sense.

- Types of truncation level 1 are *groupoids*. Identity types of groupoids are sets.

The fundamental theorem of identity types

Consider a type family $B : A \to \mathsf{Type}$ and an element $b : B(a)$. Then the following are equivalent:

1. Any family of maps
$$f : \prod_{x:A} (a = x) \to B(x)$$
is a family of equivalences.
2. The type family B is *torsorial* in the sense that
$$\sum_{x:A} B(x)$$
is contractible.
3. B is an *identity system*.

The univalence axiom

The *univalence axiom* can be stated in one of the following equivalent ways:

1. The family of maps

$$\text{equiv-eq} : \prod_{X:\mathcal{U}} (A = X) \to (A \simeq X)$$

   given by $\text{equiv-eq}(\text{refl}) := \text{id}$ is an equivalence.

2. The type

$$\sum_{X:\mathcal{U}} A = X$$

   is contractible.

3. The type family $X \mapsto A = X$ is an identity system on the universe $\mathcal{U}$.

The strucuture identity principle (by example)

Structure	*Sameness*
Groups/Rings/Modules	Isomorphism
Posets	Isomorphism

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Cubes/shapes/polytopes	Symmetries

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Importance

1. Establishing uniqueness results or properties.
2. Constructing extensions via descent.
3. Counting objects!

Truncations

For any truncation level k there is a universal way of k-truncating any type.

The *truncation operation*

$$\|-\|_k : \mathcal{U} \to \mathcal{U}^{\leq k} := \sum_{X:\mathcal{U}} \mathsf{is\text{-}trunc}_k(X)$$

comes equipped with a map $\eta : X \to \|X\|_k$ for every $X : \mathcal{U}$ and satisfies the *universal property of truncations*: The map

$$(\|X\|_k \to Y) \to (X \to Y)$$

is an equivalence for k-truncated type Y.

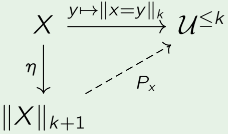
Interpretation of the truncation operations

1. The *propositional truncation* $\|X\|_{-1}$ of X is the proposition that X *is an inhabited type*.
   We say that X is *inhabited* if
   $$\text{is-contr}\|X\|_{-1}.$$
2. The *set truncation* $\|X\|_0$ of X is the *set of connected components of* X.
   We say that X is *connected* if
   $$\text{is-contr}\|X\|_0.$$
3. The 1-*truncation* $\|X\|_1$ of X is the *groupoid of simply connected components*.
   We say that X is *simply connected* if
   $$\text{is-contr}\|X\|_1.$$
4. ...and so on.

Characterizing identity types of truncations

Consider a truncation level k and an element $x : X$.

1. By the univalence axiom it follows that $\mathcal{U}^{\leq k}$ is a $(k+1)$-truncated type.
2. By the universal property of $(k+1)$-truncation, it follows that there is a unique extension



3. The family P_x is an identity system on $\|X\|_{k+1}$.
4. We have equivalences

$$\|x = y\|_k \simeq (\eta(x) = \eta(y))$$

for any $y : X$.

The premise of the Symmetry book

1. When we define a certain structure or concept, then we obtain the type of all objects of that concept.
2. By the univalence axiom it follows that symmetries of such objects are the identifications.
3. Concrete presentations of groups are therefore (set-level) structures or concepts, such that the "standard" object has the expected symmetry group.
4. All pointed connected 1-types can be obtained in this way.
5. Therefore, concrete groups are pointed connected 1-types.

Theorem

The category of groups is equivalent to the category of pointed connected 1-types.

We can do group theory with pointed connected 1-types!

The concept of being an n-element type

For a natural number n, we say that a type X *has n elements* if there is a *mere equivalence*

$$\|\mathsf{Fin}(n) \simeq X\|.$$

We define the type of all n-element types

$$BS_n := \sum_{X:\mathcal{U}} \|\mathsf{Fin}(n) \simeq X\|.$$

This is the connected component of the universe $\mathcal{U}$ at $\mathsf{Fin}(n)$. By univalence, it follows that

$$\Omega BS_n := (\mathsf{Fin}(n) = \mathsf{Fin}(n)) \simeq (\mathsf{Fin}(n) \simeq \mathsf{Fin}(n)).$$

In other words, the set ΩBS_n is the n-th symmetric group S_n.

In other words: *The type of n-element types is a delooping of the n-th symmetric group S_n.*

The concept of being an n-cycle

Consider the successor function $s : \mathsf{Fin}(n) \to \mathsf{Fin}(n)$. We say that a type X equipped with an endofunction $f : X \to X$ is an *n-cycle* if there is a mere endomorphism-preserving equivalence

$$\mathsf{is\text{-}cycle}_n(X, f) := \left\| \sum\nolimits_{e : \mathsf{Fin}(n) \simeq X} e \circ s \sim f \circ e \right\|$$

The type of all n-cycles is defined as

$$BC_n := \sum_{X : \mathcal{U}} \sum_{f : X \to X} \mathsf{is\text{-}cycle}_n(X, f).$$

By the univalence axiom it follows that there is a group isomorphism

$$\Omega BC_n := ((\mathsf{Fin}\, n, s) = (\mathsf{Fin}\, n, s)) \cong C_n,$$

where C_n is the n-th cyclic group.

The type BC_n is the delooping of the n-th cyclic group C_n.

Higher groups

Definition

A **higher group** G consists of a pointed connected type BG.

- The type BG is called the **delooping** of G.
- The base point of BG is called the **shape** of G.
- The **underlying type** of G is defined to be ΩBG. We often write G for ΩBG.

Definition

A G-**action** on a type X consists of a type family $Y : BG \to \mathrm{Type}$ equipped with an equivalence $e : X \simeq Y(*)$. Given such a G-action on X, we define the **action** μ of G on X such that $\mu(g) : X \to X$ is the unique map equipped with a homotopy

$$\begin{array}{ccc} X & \xrightarrow{e} & Y(*) \\ {\scriptstyle \mu(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_G(g)} \\ X & \xrightarrow[e]{} & Y(*). \end{array}$$

Orbits

Consider a G-action $X : BG \to \mathcal{U}$. The type of *orbits* of X is the total space

$$\sum_{u:BG} X(u)$$

Given two elements $x, y : X(\mathsf{sh}_G)$, we have an equivalence

$$((\mathsf{sh}_G, x) = (\mathsf{sh}_G, y)) \simeq \sum_{g:G} g \cdot x = y$$

An *unordered pair* of elements of A is an orbit in the $\mathbb{Z}/2$-action on A^2:

$$\text{unordered-pair}(A) := \textstyle\sum_{X:BS_2} A^X.$$

Fixed points

Consider a G-action $X : BG \to \mathcal{U}$. The type of *fixed points* of X is the function space

$$\prod_{u:BG} X(u)$$

Given a fixed point $f : \prod_{u:BG} X(u)$ and a group element $g : G$, we have the *dependent action on identifications*

$$g \cdot f(x) = f(x).$$

The type

$$\prod_{X:BS_2} X \to X$$

has two elements, corresponding to the two fixed points of the S_2-action of $\mathbb{Z}/2$ on the function type $\mathrm{Fin}(2) \to \mathrm{Fin}(2)$ given by $e, f \mapsto e \circ f \circ e^{-1}$.

Free actions

A group action $X : BG \to \mathcal{U}$ is said to be *free* if the map

$$g \mapsto g \cdot x : G \to X(\mathsf{sh}_G)$$

is an embedding for any element $x : X(\mathsf{sh}_G)$.

A group action is free if and only if its type of orbits is a set.

Binomial types

The S_k action on the type $\mathrm{Fin}\,k \hookrightarrow_{\mathsf{d}} \mathrm{Fin}\,n$ of decidable embeddings is free: The type

$$\binom{\mathrm{Fin}\,n}{\mathrm{Fin}\,k} := \sum_{X:BS_k} X \hookrightarrow_{\mathsf{d}} \mathrm{Fin}\,n$$

is equivalent to $\mathrm{Fin}\binom{n}{k}$.

Regensburg extension of the fundamental theorem of identity types (R)

Consider a subuniverse $P : \mathcal{U} \to \mathsf{Prop}$, and a pointed connected type A equipped with a family B over A. The following are equivalent:

1. Every family of maps
$$f : \prod_{x:A} (* = x) \to B(x)$$
is a family of P-maps, i.e., a family of maps with fibers in P.
2. The total space
$$\sum_{x:A} B(x)$$
is P-separated, i.e., its identity types are in P.

Transitive actions

A group action $X : BG \to \mathcal{U}$ is said to be *transitive* if the map

$$g \mapsto g \cdot x : G \to X(\mathsf{sh}_G)$$

is surjective for any element $x : X(\mathsf{sh}_G)$.

A group action is transitive if and only if its type of orbits is connected.

Torsor

A torsor is a group action $X : BG \to \mathcal{U}$ that is both free and transitive, i.e., it is a group action X such that the type of orbits

$$\sum_{u:BG} X(u)$$

is contractible.

Involutive types

An *involutive type* is a type equipped with a $\mathbb{Z}/2$-action:

$$\text{Involutive-Type} := BS_2 \to \mathcal{U}$$

The type A^2 is involutive: its $\mathbb{Z}/2$-action is given by $X \mapsto A^X$. The type of orbits of this action is the type of unordered pairs of elements in A.

For any $x, y : A$, the identity type $x = y$ is involutive.

Given an unordered pair $p := (I, a)$ of elements of A, we define the *symmetric identity type*

$$\tilde{\mathrm{Id}}(p) := \sum_{x:A} \prod_{i:I} x = a_i.$$

A *symmetric binary relation* on a type A is a family of types indexed by the unordered pairs in A:

$$\text{unordered-pair}(A) \to \mathcal{U}.$$

Adjunctions of symmetric binary relations

Adjunctions

For any binary relation $R : A \to A \to \mathcal{U}$ we can define:

1. The symmetric core:

$$\mathrm{core}(R, (I, a)) := \prod_{i:I} R(a_i, a_{-i})$$

   For any symmetric binary relation S on A we have an equivalence

$$\Big(\prod_{x,y:A} S\{x, y\} \to R(x, y)\Big) \simeq \Big(\prod_{p:\text{unordered-pair}(A)} S(p) \to \mathrm{core}(R, p)\Big).$$

2. The symmetrization:

$$\tilde{R}(I, a) := \sum_{i:I} R(a_i, a_{-i})$$

   For any symmetric binary relation S on A we have an equivalence

$$\Big(\prod_{p:\text{unordered-pair}(A)} \tilde{R}(p) \to S(p)\Big) \simeq \Big(\prod_{x,y:A} R(x, y) \to S\{x, y\}\Big)$$

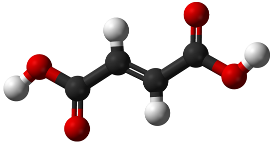
Univalent organic chemistry

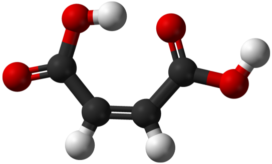
Isomerism

There are molecules that have the same underlying graph, but nevertheless we can distinguish them based on their spatial arrangement. Such pairs of molecules are called **isomers**.

Question

How do we define hydrocarbons in univalent mathematics in such a way that distinct isomers are correctly distinguished?





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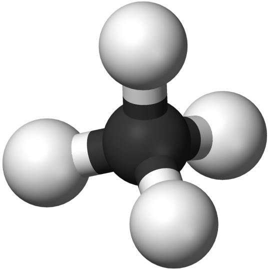
The basic idea in the definition of the hydrocarbons

Main points

- A hydrocarbon consists of hydrogen atoms and carbon atoms.
- Hydrogen atoms form exactly one bond.
- Carbon atoms form exactly four bonds.
- To account for the spatial arrangement of a hydrocarbon, we must restrict the symmetry group of each carbon atom:
  - Any symmetry that fixes one point, must preserve the cyclic ordering of the remaining three points.
  - Any symmetry that fixes two points also fixes the remaining two points.

  In other words, the symmetry group of a carbon atom in 3-space is A_4.



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Undirected graphs

Definition

An **(undirected) graph** $G \doteq (V, E)$ consists of

- A type V of **vertices**
- A type family

  $$E : \text{unordered-pair}(V) \to \mathcal{U}$$

  of **half-edges**, indexed by unordered pairs.

Remark

- If $e : E(\{x, y\})$ is a half-edge, we think of it as the half-edge starting at x pointing in the direction of y. We have $E(\{x, y\}) \simeq E(\{y, x\})$ which relates the two halves of an edge.
- Graphs can have multiple half-edges between vertices.
- Graphs can have loops (half-edges from a vertex pointing to itself).
- We don't make any restriction on the truncation levels of the vertices or the edges.

Two graphs with one vertex

Example

Consider the graph with

$$V := \mathbf{1}$$
$$E(X,f) := \mathbf{1}$$

Then $E(\{*,*\}) := \mathbf{1}$ so this graph has one half-edge. The total space of all edges is

$$\sum_{(X,f)} E(X,f) \simeq BS_2.$$

There is indeed only half an edge! This graph looks like

Example

Consider the graph with

$$V := \mathbf{1}$$
$$E(X,f) := X$$

Then $E(\{*,*\}) := \mathrm{Fin}(2)$ so this graph has two half-edges. The total space of all edges is

$$\sum_{(X,f)} E(X,f) \simeq \mathbf{1}.$$

This graph has a loop on the unique vertex. This graph looks like

Enriched undirected graphs

Definition

Consider an undirected graph $G \doteq (V, E)$, and let $v : V$. Then we define the **neighborhood** of v by

$$\mathrm{Neighborhood}_G(v) := \sum_{(x:V)} E(\{v, x\})$$

Definition

Consider a type A and a type family B over A. An (A, B)**-enriched undirected graph** consists of

- An undirected graph $G \doteq (V, E)$.
- A map $\mathrm{sh} : V \to A$. We call $\mathrm{sh}(v)$ the **shape** of v.
- For each vertex $v : V$ an equivalence

$$e_v : B(\mathrm{sh}(v)) \simeq \mathrm{Neighborhood}_G(v).$$

∞-group actions of (A, B)-enriched graphs

Remark

Consider an (A, B)-enriched graph (V, E, sh, e).

- For every vertex $v : V$ we obtain an ∞-group

$$BG_v := \sum_{(x:A)} \|\mathrm{sh}(v) = x\|.$$

  Its base point is $\mathrm{sh}(v)$, and we write G_v for its underlying type. G_v is called the **symmetry group** of v.

- For every vertex $v : V$ we obtain a G_v-type

$$B : BG(v) \to \mathcal{U}$$

  given by restricting B.

- By the equivalence $B(\mathrm{sh}(v)) \simeq \mathrm{Neighborhood}(v)$ we obtain an action

$$G_v \to (\mathrm{Neighborhood}(v) \to \mathrm{Neighborhood}(v))$$

  of the symmetry group of v on its neighborhood.

Equivalences of (A, B)-enriched graphs

Definition

An equivalence between two (A, B)-enriched graphs (V, E, sh, e) and $(V', E', \mathrm{sh}', e')$ consists of

- An equivalence $\alpha : V \simeq V'$
- A family of equivalences $\beta : E(p) \simeq E'(\text{unordered-pair}(\alpha, p))$ for each $p : \text{unordered-pair}(V)$
- A homotopy $\gamma : \mathrm{sh} \sim \mathrm{sh}' \circ \alpha$
- For each vertex v a commuting square

$$\begin{array}{ccc}
B(\mathrm{sh}(v)) & \xrightarrow{\mathrm{tr}_B(\gamma(v))} & B(\mathrm{sh}'(\alpha(v))) \\
{\scriptstyle e(v)}\downarrow & & \downarrow{\scriptstyle e(\alpha(v))} \\
\mathrm{Neighborhood}_{(V,E)}(v) & \xrightarrow[\mathrm{Neighborhood}_{(\alpha,\beta)}(v)]{} & \mathrm{Neighborhood}_{(V',E')}(\alpha(v))
\end{array}$$

Equivalences of (A, B)-enriched graphs are shape-preserving equivalences of graphs that also preserve the action of the symmetry group of a vertex on its neighborhood.

The sign homomorphism

Theorem (Mangel, R.)

For any $n : \mathbb{N}$ there is a map $\sigma : BS_n \to BS_2$ such that the square of group homomorphisms

$$\begin{array}{ccc} S_n & \xrightarrow{\text{sign}} & S_2 \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \Omega BS_n & \xrightarrow[\Omega\sigma]{} & \Omega BS_2 \end{array}$$

commutes.

Remark

The construction of σ involves:

- A functorial construction that turns an arbitrary n-element set X into a 2-element set $\sigma(X)$.
- As a functor, $\sigma : BS_n \to BS_2$ must be full (surjective on morphisms).
- In HoTT we say that σ must be 0-connected.

The alternating groups

Definition

We define BA_n as the pullback

$$\begin{array}{ccc} BA_n & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ BS_n & \xrightarrow[\sigma]{} & BS_2. \end{array}$$

Definition

- Define the type of **hydrogen atoms** to be

$$\mathcal{H} := \mathbf{1}.$$

  Note that there is a canonical map

$$\gamma_{\mathcal{H}} : \mathcal{H} \to BS_1.$$

- Define the type of **carbon atoms** to be

$$\mathcal{C} := BA_4,$$

  where BA_4 is the classifying type of the alternating group A_4. Note that there is a canonical map

$$\gamma_{\mathcal{C}} : \mathcal{C} \to BS_4.$$

Definition

The **type of hydrocarbons** is defined to be the type of (A, B)-enriched graphs where

$$A := \mathcal{H} + \mathcal{C} \qquad : \mathcal{U}$$
$$B := [\gamma_{\mathcal{H}}, \gamma_{\mathcal{C}}] \qquad : A \to \mathcal{U}$$

such that

- The type of vertices is finite.
- The underlying graph is connected.
- The underlying graph has no loops.

We developed an intuition in univalent mathematics that allowed us to interpret common concepts as structured types.

However: *All our definitions are correct interpretations of concepts in any structural foundation of mathematics based on dependent type theory. Univalence is not necessary to justify our definitions.*